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# Use Of Fourier Transform In The Theory Of Finance

**Jigna V. Panchal**

Indus Institute of Technology and Engineering,Indus University, Rancharada.

jignapach@gmail.com

*Corresponding author

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**Abstract**

In Financial world, option is one of the definite types of contracts. Basically there are two types of option, Call and Put option. The important part is the valuation of this option and it is represented by a well-known Black-Scholes-Merton (BSM) Partial Differential Equation.It is of second order parabolic type Partial Differential Equation. The solution of this equation gives the theoretical value of an option (Call/Put). Here in the present paper we have solved BSM equation with the help of Fourier Transform. In fact we have solved BSM equation for two different Payoff functions: Standard Power option and Powered option as boundary conditions.

Keywords: Black-Scholes-Merton Model; Partial Differential Equation; Fourier Transform Method; Call/Put option.

## Introduction

The pricing of option is very important problem in financial market.In option pricing theory, the Black-Scholes-Merton equation is one of the most effective models for pricing options. If we consider the European call option, which gives the right to buy an asset on a specific future date, at a specific price,which depends on S-Spot price, X-Exercises Prices, t-Expiration date, r-risk free interest rate, and $\sigma$-Volatility.This model is very useful, since this requires five variables only in which four are easily available, those are  and for the volatility we have to use historical data to estimate it.The formula developed by three economists- Fisher Black, Myron Scholes and Robert Merton. They were awarded the 1997 Nobel Prize in economics for their work [1].

## Theoretical Analysis

In Mathematical Finance, the Black-Scholes-Merton equation is a Partial Differential Equation to evaluate the value of European Call/put option. Suppose C(S,t) is the call premium then the equation [3], [5], [7],

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$$

is a Black-Scholes-Merton Partial Differential Equation.

Where,

$$t\in[0,T]\text{ and } C(0,t)=0 \text{ for all } t \text{ and } C(S,t)\to S \ as \ S\to\infty$$

Consider the European call option whose final payoff at the expiry time *T* is given by a function $f$ of the final spot price S. $\lim_{t\to T_-} C(S,t)=f(S)$ which is a continuous function that need not be differentiable everywhere.

The common assumptions of the BSMare [7]:

(1) The options are European type
(2) No dividends are paid
(3) Movement of the market cannot be predicted
(4) No Commission and no transaction cost
(5) Interest rates are risk free and it is constant
(6) Volatility is constant
(7) Returns are normally distributed.

To solve the above mathematical model various approaches and methods are proposed by researcher like the Method of Separation of Variable, Method of Laplace Transform [2]. Some of them having limitations and some having advantages over them. In this paper we have applied Fourier Transform Method to solve the problem. Our goal is to find the function $C(S,t):(0,\infty)\times[0,T]\to[0,\infty)$, which satisfy the given Partial Differential Equation. This equation has many solutions corresponding to all the different functions that can be defined withs as the underlying variable. The particular value that is obtained when the equation is solved depends on the Payoff functions. These specify the values of the call option at the boundaries of possible values of and . Here we will discuss two different Payoff functions: Standard Power option and Powered option. First, we will convert the Black-Scholes-Merton Partial Differential Equation to the heat equation by the following substitutions:

$$y=T-t,$$

$$x=\ln\left(\frac{S}{X}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)\text{ and}$$

$$D(x,y)=e^{r(T-t)}C(S,t).$$

These substitutions also convert the above mentioned boundary condition,

$$\lim_{t \to T_-} C(S,t) = f(S)$$

into the initial condition,

$$\lim_{y \to 0^+} D(x,y) = f(Xe^x)$$

Thus the Black-Scholes-Merton equation gets converted into the following Heat Equation with the stated initial condition:

$$\frac{\partial D}{\partial y} = \frac{\sigma^2}{2}\frac{\partial^2 D}{\partial x^2} \quad \text{with} \quad \lim_{y \to 0^+} D(x,y) = f(Xe^x) \quad (1)$$

Applying Fourier transform on both the sides of the equation (1) we get,

$$\frac{\partial}{\partial y} F(D) + \frac{\lambda^2 \sigma^2}{2} F(D) = 0$$

$$\therefore \quad F(D) = C_1 e^{\frac{-\sigma^2 \lambda^2}{2} y}$$

Now we get,

$$F(D(x,0)) = G(\lambda) \quad \text{because} \quad D(x,0) = f(Xe^x)$$

Here, is the Fourier transform of , so that is determined and we now have:

$$F(D) = G(\lambda) e^{\frac{-\sigma^2 \lambda^2}{2} y}$$

Taking inverse Fourier transform on both the sides we get,

$$D(x,y) = F^{-1}\left( G(\lambda) e^{\frac{-\sigma^2 \lambda^2}{2} y} \right)$$

Now, using convolution theorem and from the facts,

$$F^{-1}(G(\lambda)) = f(Xe^x) \quad \text{and} \quad F^{-1}\left( e^{\frac{-\sigma^2 \lambda^2}{2} y} \right) = \frac{1}{\sigma\sqrt{y}} e^{\frac{-x^2}{2\sigma^2 y}}$$

we get:

$$D(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\frac{1}{\sigma\sqrt{y}}e^{\frac{-(x-v)^2}{2\sigma^2 y}}dv$$

$$\therefore\ D(x,y)=\frac{1}{\sigma\sqrt{2\pi y}}\int_{-\infty}^{\infty} f(v)e^{\frac{-(x-v)^2}{2\sigma^2 y}}dv$$

This is the required solution ofthe Heat equation [6].

**Solution Of The Problem Using Different Payoff Functions**

Many people have solved BSM equation for Standard Power Payoff function and Powered Payoff function using the Method of Separation of Variable [4].

**Standard Power Payoff Function**

Consider the Payoff function, which is known as Standard Power Payoff function:

$$=\frac{X^2}{\sigma\sqrt{2}}$$

$$\therefore D(x$$

Taking,

$$Z=\frac{v-x}{\sigma\sqrt{y}}$$

We get,

$$\therefore D(x,y)=\frac{X^2}{\sqrt{2\pi}}e^{2(x+\sigma^2 y)}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z^2-4\sigma\sqrt{y}Z+4\sigma^2 y)}{2}}dZ-\frac{X}{\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}}dZ$$

$$=\frac{X^2}{\sqrt{2\pi}}e^{2(x+\sigma^2 y)}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z-2\sigma\sqrt{y})^2}{2}}dZ-\frac{X}{\sqrt{2\pi}}\int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}}dZ$$

$$= \frac{X^2}{\sqrt{2\pi}} e^{2(x+\sigma^2 y)} \int_{-\frac{x+2\sigma^2 y}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-t^2}{2}} dt - \frac{X}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}} dZ$$

$$= \frac{X^2}{\sqrt{2\pi}} e^{2(x+\sigma^2 y)} \int_{-\infty}^{\frac{x+2\sigma^2 y}{\sigma\sqrt{y}}} e^{\frac{-t^2}{2}} dt - \frac{X}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{\frac{-t^2}{2}} dt$$

$$= X^2 e^{2(x+\sigma^2 y)} N(d_1) - XN(d_2)$$

Where,

$$d_1 = \frac{x+2\sigma^2 y}{\sigma\sqrt{y}} \ , \ d_2 = \frac{x}{\sigma\sqrt{y}} \ \ and \ \ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$

$$\therefore C(S,t) = S^2 e^{\left(r+\sigma^2\right)(T-t)} N(d_1) - Xe^{-r(T-t)} N(d_2)$$

Where,

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{3\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \ \ and \ \ d_2 = d_1 - 2\sigma\sqrt{T-t}$$

**Powered Payoff Function**

Now, we consider the Payoff function, which is known as Powered Payoff function:

$$f(S) = \max\{(S-X)^2, 0\}$$

$$\Rightarrow f(Xe^x) = \max\{X^2(e^x-1)^2, 0\}$$

$$\therefore D(x,y) = \frac{1}{\sigma\sqrt{y}} \int_0^{\infty} X^2 (e^v - 1)^2 e^{\frac{-(x-v)^2}{2\sigma^2 y}} dv$$

$$= \frac{X^2}{\sigma\sqrt{y}} \left[ \int_0^{\infty} e^{2v} e^{\frac{-(x-v)^2}{2\sigma^2 y}} dv - 2\int_0^{\infty} e^{v} e^{\frac{-(x-v)^2}{2\sigma^2 y}} dv + \int_0^{\infty} e^{\frac{-(x-v)^2}{2\sigma^2 y}} dv \right]$$

Taking,

$$Z = \frac{v - x}{\sigma\sqrt{y}}$$

We get,

$$D(x,y) = X^2 e^{2(x+\sigma^2 y)} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z^2-4\sigma\sqrt{y}Z+4\sigma^2 y)}{2}} dZ - 2X^2 e^{x+\frac{\sigma^2 y}{2}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z^2-2\sigma\sqrt{y}Z+\sigma^2 y)}{2}} dZ + X^2 \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}} dZ$$

$$= \frac{X^2}{\sqrt{2\pi}} e^{2(x+\sigma^2 y)} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z-2\sigma\sqrt{y})^2}{2}} dZ - \frac{2X^2}{\sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-(Z-\sigma\sqrt{y})^2}{2}} dZ + \frac{X^2}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}} dZ$$

$$= \frac{X^2}{\sqrt{2\pi}} e^{2(x+\sigma^2 y)} \int_{-\frac{x+2\sigma^2 y}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-t^2}{2}} dt - 2\frac{X^2}{\sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\frac{x+\sigma^2 y}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-t^2}{2}} dt + \frac{X^2}{\sqrt{2\pi}} \int_{-\frac{x}{\sigma\sqrt{y}}}^{\infty} e^{\frac{-Z^2}{2}} dZ$$

$$= \frac{X^2}{\sqrt{2\pi}} e^{2(x+\sigma^2 y)} \int_{-\infty}^{\frac{x+2\sigma^2 y}{\sigma\sqrt{y}}} e^{\frac{-t^2}{2}} dZ - \frac{2X^2}{\sqrt{2\pi}} e^{x+\frac{\sigma^2 y}{2}} \int_{-\infty}^{\frac{x+\sigma^2 y}{\sigma\sqrt{y}}} e^{\frac{-t^2}{2}} dt + \frac{X^2}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma\sqrt{y}}} e^{\frac{-t^2}{2}} dt$$

$$= X^2 e^{2(x+\sigma^2 y)} N(d_1) - 2X^2 e^{x+\frac{\sigma^2 y}{2}} N(d_2) + X^2 N(d_3)$$

where,

$$d_1 = \frac{x+2\sigma^2 y}{\sigma\sqrt{y}} \ , \ d_2 = \frac{x+\sigma^2 y}{\sigma\sqrt{y}} \ , \ d_3 = \frac{x}{\sigma\sqrt{y}} \ and \ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$

where,

$$\therefore C(S,t) = S^2 e^{(r+\sigma^2)(T-t)} N(d_1) - 2SXN(d_2) + X^2 e^{-r(T-t)} N(d_3)$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{3\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \ , \ d_2 = d_1 - \sigma\sqrt{T-t}, \ and \ d_3 = d_1 - 2\sigma\sqrt{T-t}$$

## Conclusion

The BSM equation is solved using Fourier Transform Method. This gives the value of an option for the above mentioned two different Payoff functions. Using this solution the trader can find the theoretical value of options (call/put) on a verity of assets including securities, commodities, currencies etc. with stipulated Payoff functions. It is hoped that in certain situations solution found will be useful. Also the method used here has scope of wider adoption.

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